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# Eigenvalue bounds in magnetoatmospheric shear flow

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**Abstract.** A rigorous approach by Barston to stability of Lagrangian systems is used to establish both rectangle and semicircle theorems for plane parallel flow along a horizontal but otherwise arbitrary magnetic field, permeating a perfectly electrically conducting incompressible fluid under gravity. The radius of the semicircle is reduced by magnetic effects and stable stratification. A Richardson criterion for stability against constant shear flow is also derived. The analogous problem for a compressible fluid is also discussed, and for a certain class of disturbances a 'semi-dumbell' theorem is established which is considerably stronger than the semicircle theorem. Possible astrophysical applications are discussed.

## 1. Introduction

Since the appearance of a paper by Howard (1961) on eigenvalue bounds for plane parallel flow of an inviscid incompressible stratified fluid, many subsequent papers extending and generalising the results have been published (see e.g. Eckart 1963, Agrawal 1969, Chimonas 1970, Acheson 1973, Adam 1978b, among many others). In this paper we firstly make use of very general theorems developed by Barston (1977) in a study of eigenvalue problems for Lagrangian systems, to investigate eigenvalue bounds for magnetogravity shear flow, that is Howard's original problem with a horizontal non-uniform magnetic field superimposed upon the flow. The stability of rotating magnetic fluids has received much attention in the past (seeAcheson 1973 for further references) and therefore the semicircle theorem and rectangle theorem obtained here for plane parallel flow are perhaps not surprising. Nevertheless, the rigorous formulation of the problem in the first part of the paper leads to and justifies the second aspect of this work, namely consideration of the more difficult eigenvalue problem associated with the *compressible* situation, that is magnetoatmospheric flow. This topic has been considered by the author elsewhere (Adam 1978a, hereafter referred to as I). A number of results were then established based on the concept of the complex 'wave-energy flux' function, including a semicircle theorem, but the effective radius of the semicircle, while elucidating the stabilising/destabilising effects of the various restoring forces, depends on phase velocity. This result is clearly not optimal and the second part of this paper contains an improvement of the theorem, subject to a certain constraint. This result, a 'semi-dumbell' theorem (see figure 2) is considerably stronger than the semicircle theorem previously derived (for a certain class of disturbance) in that it reduces the régime of existence of complex (and hence unstable) magnetoatomospheric shear modes. The term 'semi-' refers to the imaginary part $c_i$ of

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the complex phase velocity $c$ for which $c_i>0$; obviously since the problem here is an idealised non-dissipative one, for every $c_i>0$ there exists a $-c_i<0$ corresponding to damped oscillatory (for $c_r \neq 0$) disturbances.

In § 2 we give the basic background and theorems necessary for the application to magnetogravity flow in § 3. We consider the general magnetoatmospheric case in § 4, and show how in the incompressible limit the rigorous results in § 3 are recovered. Section 5 contains the semi-dumbell theorem, giving a bounding curve containing the complex eigenvalues corresponding to unstable magnetoatomospheric modes, subject to a certain constraint. This constraint is examined in terms of the upper and lower bounds on the flow velocity, and brief solar-physical applications are discussed. It is hoped to present these in more detail elsewhere.

## 2. Theoretical formulation

In this section we state without proof the relevant theorems derived by Barston (1977), to be applied to the governing differential equations (7) and (8) in § 3. For further details the reader is urged to consult Barston (1977).

The linearised equations governing small perturbations $\xi$ about a state of steady motion of a conservative dynamical system can be expressed in the following canonical form (see Barston (1977) for a list of further references)

$$P\ddot{\xi}+A\dot{\xi}+H\xi=0, \qquad \xi=\xi(x, z; t) \tag{1}$$

where $P$, i$A$ and $H$ are time-independent, linear, formally self-adjoint operators with domains of definition $D_P$, $D_A$, $D_H$ respectively and range in an inner product space $E$. The above domains are all linear subspaces of $E$. By seeking solutions of the form $\xi=\zeta \exp(\mathrm{i}\omega t)$ where $\zeta \neq 0 \in E$, $\omega \in \mathbb{C}$, we obtain the quadratic eigenvalue problem

$$(\omega^2 P-\omega \mathrm{i}A-H)\zeta=0. \tag{2}$$

By the term 'formally self-adjoint' applied to an operator $F$ in $E$ we mean $(\eta, F\zeta)=(F\eta, \zeta)$ for all $\eta, \zeta \in D_F$,

$$(\eta, F\zeta)=\int_{D_F} \bar{\eta}F\zeta \, \mathrm{d}z$$

($\bar{\ }\equiv$ complex conjugate). Let $P>0$ on $D_p$. For all non-zero $\zeta \in D=D_p \cap D_A \cap D_H$ we define the following real functions of a complex variable:

$$H^*(\zeta)=(\zeta, H\zeta) \qquad P^*(\zeta)=(\zeta, P\zeta) \qquad A^*(\zeta)=(\zeta, \mathrm{i}A\zeta) \tag{3}$$

$$\bar{H}(\zeta)=H^*(\zeta)/P^*(\zeta), \qquad \bar{A}(\zeta)=A^*(\zeta)/P^*(\zeta). \tag{4}$$

Let $d(\zeta)=\bar{H}^2/4+\bar{H}$ and $Q_\pm(\zeta)=\bar{A}/2+d^{1/2}$. Then $\zeta \in D$ ($\omega \neq 0$) and $H_\omega\zeta=0$, where $H_\omega \equiv H+\omega \mathrm{i}A-\omega^2 P$, $H_\omega$ is a linear operator from $D$ into $E$, formally self-adjoint on $D$ (for $\zeta \in R$).

Hence $(\zeta, H_\omega\zeta)=H^*_\omega(\zeta)=0$, that is

$$\omega^2 P^*-\omega A^*-H^*=0 \tag{5}$$

which implies $\omega=Q_+(\zeta)$ or $\omega=Q_-(\zeta)$. Define $\Delta \equiv \inf_D d(\eta)$ (the infimum being taken over all non-zero elements of $D$). We now state three preliminary lemmas necessary for the theorems we shall state and use below. In what follows $\zeta$ will be used to represent a general point in $D$, while $\eta$ will refer to points in the subset $\tilde{D}$ for which $d(\eta)<0$.

*Lemma 1.* If $\omega$ is an eigenvalue of equation (1) with corresponding eigenvector $\zeta$, and $\text{Im}\,\omega \neq 0$, then $\zeta \in D$ and

$$|\omega|^2 = \bar{H}(\zeta) \leq \sup_{\tilde{D}}[-H(\eta)] \equiv s^2$$

where the set $\tilde{D} = \{\eta: \eta \in D, \eta \neq 0, d(\eta) < 0\}$ is non-empty.

This result means that the complex eigenvalues of equation (1) all lie within a circle of radius $s$ centred at the origin of the complex $\omega$-plane. This is the type of result we seek, but the theorem below expresses a more useful bound on the eigenvalues.

*Lemma 2.* A necessary and sufficient condition that the system (1) possess a complex eigenvalue $\omega$ is that $\Delta < 0$ (see equation (4))

*Lemma 3.* A necessary condition that $\Delta < 0$ is the existence of a $\phi \in D$ such that $(\phi, -H_\alpha \phi) > 0$ for all real $\alpha$.

We now suppose that $\Delta < 0$ and define the radius function $\lambda^{1/2}(\alpha)$, for all real $\alpha$, where

$$\lambda(\alpha) \equiv \{\sup_{\tilde{D}}[-H_\alpha(\phi)]\} \tag{6}$$

and that $r$ is a real-valued function such that $r(\alpha) \geq \lambda^{1/2}(\alpha)$ for all real $\alpha$.

We now state the most relevant parts of Barston's work in the following theorem:

*Theorem 1.* (i) Every non-real eigenvalue $\omega$ of the system (1) lies inside the circle $|y-\alpha| \leq r(\alpha)$ in the Argand plane with $\alpha$ real.

(ii) $r(\alpha) \geq (-\Delta)^{1/2}$, $\alpha \in (-\infty, \infty)$.

(iii) (Semicircle theorem). Let $C_\beta(r)$, $\beta \in R$ be the circle of minimum radius for the given majorising function $r$. Then every eigenvalue $\omega$ of equation (1) with $\text{Im}\,\omega < 0$ lies in the semicircle $C_\beta(r) \cap \{y: y \in C, \text{Im}\, y < 0\}$.

(iv) (Rectangle theorem). Every non-real eigenvalue of equation (1) lies inside the rectangle $M$ given by $|\text{Im}(y)| \leq (-\Delta)^{1/2}$, $\inf \bar{A}(\eta) \leq 2\,\text{Re}(y) \leq \sup \bar{A}(\eta)$ in the Argand plane where the $\eta$'s are chosen from $\tilde{D}$.

It can be shown, as indicated in part (ii), that the least radius of the circle is $(-\Delta)^{1/2}$.

## 3. Application to magnetogravity flow

For this problem we state the governing linearised coupled equations for the problem. The case of magneto-atmospheric flow has been discussed in detail in I (see equations (A1) and (A2) in the Appendix) and we take the ($\gamma \to \infty$) incompressible limit to obtain

$$\frac{\partial p}{\partial z} + \rho_0\left(n_0^2 + \frac{\mathrm{d}^2}{\mathrm{d}t^2} - a_0^2 \frac{\partial^2}{\partial x^2}\right)\xi = 0 \tag{7}$$

$$\frac{-\partial^2 p}{\partial x^2} + \rho_0\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} - a_0^2 \frac{\partial^2}{\partial x^2}\right)\frac{\partial \xi}{\partial z} = 0. \tag{8}$$

The notation is as follows: $p = p_g + (\boldsymbol{H}_0 \cdot \boldsymbol{h}/4\pi)$ is the total pressure perturbation due to compressive and magnetic effects, $\xi$ is the vertical component of the Lagrangian displacement vector, $\rho_0(z)$ is the equilibrium density distribution, $a_0 = H_0/(4\pi\rho_0)^{1/2}$ is the Alfvén velocity, $n_0^2 = (-g/\rho_0)(\mathrm{d}\rho_0/\mathrm{d}z)$ where $\boldsymbol{g} = (0, 0 - g)$ is the acceleration due to gravity. $(\mathrm{d}/\mathrm{d}t) \equiv \partial/\partial t + U_0(z)\partial/\partial x$ for flow $\boldsymbol{U} = (U_0(z), 0, 0)$. Let

$$\begin{pmatrix} p \\ \xi \end{pmatrix} = \int_{-\infty}^{\infty} \exp(\mathrm{i}kx)\begin{Bmatrix} \hat{p} \\ \hat{\xi} \end{Bmatrix} \mathrm{d}k$$

denote the respective space Fourier transform relationships. From now on we deal with $\hat{p}$, $\hat{\xi}$. Eliminating $\hat{p}$ using equation (8) we obtain the equation

$$k^2\rho_0\left[n_0^2 + a_0^2k^2 + \left(\frac{\partial}{\partial t} + \mathrm{i}kU_0\right)^2\right]\hat{\xi} - \frac{\partial}{\partial z}\left[\rho_0\left(\frac{\partial}{\partial t} + \mathrm{i}kU_0\right)^2 \frac{\partial \hat{\xi}}{\partial z}\right] - \frac{\partial}{\partial z}\left[\frac{H_0^2}{4\pi}k^2 \frac{\partial \hat{\xi}}{\partial z}\right] = 0. \tag{9}$$

This can be arranged in the form (1) (dropping the circumflex)

$$P\ddot{\xi} + A\dot{\xi} + H\xi = 0 \tag{10}$$

where the expressions for $P$, $A$ and $H$ are easily derived. We are concerned with the layer $z_1 \leqslant z \leqslant z_2$, and the Lagrangian perturbation $\xi$ is assumed to vanish at $z = z_1$ and $z = z_2$ at all times.

(The operators $P$, $A$ and $H$ map $D$ into the inner product space $E = C[z_1, z_2]$; thus $D \subset E$. It is easily shown that $P$, i$A$ and $H$ are all formally self-adjoint on $D$, and that $P$ is positive definite on $D$.) Define the symbol $\langle\ \rangle$ by

$$\langle B \rangle \equiv \int_{z_1}^{z_2} B\, \mathrm{d}z \tag{11}$$

then

$$\begin{aligned} &P^*(\phi) = \langle \tilde{Q}(\phi) \rangle \qquad \text{where } \tilde{Q}(\phi) = \rho_0[|\phi'|^2 + k^2|\phi|^2] > 0 \\ &A^*(\phi) = -2k\langle U_0\tilde{Q}(\phi) \rangle \\ &H^*(\phi) = -k^2\langle (U_0^2 - a_0^2)\tilde{Q}(\phi) \rangle - k^2\langle g\rho_0'|\phi|^2 \rangle \end{aligned} \tag{12}$$

where $k \neq 0$, $\phi \in D$, $\phi \neq 0$, $\rho_0 > 0$. Therefore

$$\begin{aligned} -H_{-kc}(\phi) &= (\overline{k^2c^2P}) + (\overline{kcA}) - \bar{H} \qquad \text{for real } c \\ &= k^2\{\langle [(U_0 - c)^2 - a_0^2]\tilde{Q} \rangle + \langle g\rho_0'|\phi|^2 \rangle\}/\langle \tilde{Q} \rangle. \end{aligned} \tag{13}$$

We define $\lambda(c) = \sup[-H_{-kc}(\phi)]$ for all real $c$.

Then it follows from the theorem that for each $c \in R$ every complex eigenvalue $\omega$ lies within the circle $C_{-kc}$ with centre $(-kc, 0)$ and radius $[\lambda(c)]^{1/2}$($\lambda(c) \leqslant 0$ for some $c$ implies all eigenvalues must be real). Let $a \leqslant U_0(z) \leqslant b$ on $[z_1, z_2]$. Then

$$\begin{aligned} \sup\{\langle (U_0 - c)^2\tilde{Q} \rangle / \langle \tilde{Q} \rangle\} &= G(c) \\ &= [|\tilde{c}| + (b - a)/2]^2 \end{aligned} \tag{14}$$

where $\tilde{c} = c-(a+b)/2$. Similarly

$$\sup\{\langle -a_0^2\tilde{Q}^2\rangle/\langle\tilde{Q}\rangle\} = -\inf\{\langle a_0^2\tilde{Q}\rangle/\langle\tilde{Q}\rangle\} = -a_{\min}^2, \text{ say.} \tag{15}$$

If

$$g\rho_0'(z) \leqslant k^2[G(c)-a_{\min}^2-\tilde{\mu}^{-1}\tilde{N}^2] \tag{16}$$

for all real $c$ where

$$\tilde{N}^2 = \min_{[z_1,z_2]}\left(-\frac{g\rho_0'(z)}{\rho_0(z)}\right) > 0$$

(where $-g\rho_0'/\rho_0$, if positive, is the square of the Brunt–Vaisala frequency, this being the frequency at which a fluid element oscillates about its position of stable equilibrium when displaced from that position) and

$$\mu = \max_{\phi\in D}\{\langle\tilde{Q}\rangle/\langle\rho_0|\phi|^2\rangle\}\rangle k^2.$$

Since the minimum value of the function $G(c)$ is $\frac{1}{4}(b-a)^2 = \frac{1}{2}G(a+b)$, this result implies that every complex eigenvalue $\omega$ lies within the circle with centre $(-k(a+b)/2, 0)$ and radius $|k|[\frac{1}{4}(b-a)^2-a_{\min}^2-\tilde{\mu}^{-1}\tilde{N}^2]$. This is a circle theorem with radius reduced compared with the non-magnetic case. Physically we may argue that in a potentially unstable situation, some energy associated with unstable modes due to shear stresses must overcome Lorentz restoring forces in 'corrugating' the magnetic field lines according to wavenumber $k$. Thus the effective growth rate of any surviving instabilities is reduced compared with the non-magnetic case.

If the physical situation is such that locally convectively unstable regions exist, that is $g\rho_0'(z)>0$ for some $z\in[z_1, z_2]$ (see Adam (1977) for a detailed discussion of the compressible non-magnetic case) then

$$\lambda_0(c) \leqslant k^2(G(c)-a_{\min}^2+\mu^{-1}N^2) \qquad \forall \text{ real } c$$

where

$$N^2 \equiv \max_{[z_1,z_2]}\left[\frac{g\rho_0'(z)}{\rho_0(z)}\right] > 0 \qquad \text{and} \qquad \mu = \min_{\phi\in D}\{\langle\tilde{Q}\rangle/\langle\rho_0|\phi|^2\rangle\} > k^2.$$

Thus every complex eigenvalue lies in the circle with centre $(-k(a+b)/2, 0)$ and radius $|k|\{\frac{1}{4}(b-a)^2-a_{\min}^2+\mu^{-1}N^2\}$.

Finally in this section, we note two further results for the present system. In § 2, theorem 1(iv) states that every complex eigenvalue $z$ of the Lagrangian system (1) lies within the rectangle $M$. Now $\tilde{A} = -2k\langle U_0\tilde{Q}\rangle/\langle\tilde{Q}\rangle$, so we obtain a rectangle theorem for complex $\omega$, with $\omega_i \leqslant (-\Delta)^{1/2}$, $-kb \leqslant \omega_r \leqslant -ka$. This rectangle is inscribed by the semicircle at maximum radius $\frac{1}{2}(b-a)$ and is obviously a weaker result. For convenience we represent the diagrams illustrating the results established in this section with those in subsequent sections, in the *positive* upper quadrant, so we note a change from $\omega$ to $-\omega$ in presenting these results. We can also obtain very simply a sufficient condition for stability for a linear flow profile (constant shear) $U_0(z)$. It is clear from equation (16) that in the case of stably-stratified magnetogravity flow a sufficient condition for stability is that $R^2 = \frac{1}{4}(b-a)^2-a_{\min}^2-\tilde{\mu}^{-1}\tilde{N}^2 \leqslant 0$. Given a constant shear flow $U_0(z)$ in $[z_1, z_2]$ we write

$$U_0(z)=\frac{b-a}{L}z+a-\frac{(b-a)}{L}z_1$$

where $L=z_2-z_1>0$. Then $a\leqslant U_0\leqslant b$ in $[z_1, z_2]$ and the above condition becomes

$$\frac{\tilde{\mu}^{-1}\tilde{N}^2+a_{\min}^2}{(LU_0')^2}\geqslant\tfrac{1}{4} \tag{17}$$

for stability. This is similar in form to the well-known Richardson criterion for the stability of plane parallel flow of an incompressible inviscid fluid (Howard 1961). Thus if, in the case of constant shear, the stabilising effects of stratification and magnetic field are sufficiently large compared with the destabilising effect of shear, the flow will be stable.

## 4. General magnetoatmospheric case

It has been shown in I that the general magnetoatmosphere shear problem sustains, among other things, a smicircle theorem for unstable modes, for the free boundary case. Here we consider the *rigid* boundary case in order to compare with the results of the previous section (by taking the limit of incompressibility). We also obtain an improvement—subject to a certain constraint on the eigenvalue bound for the compressible case.

The basic equations used in I are given in the Appendix. Here we merely state the results. The governing coupled equations are, in operator form

$$\begin{pmatrix} d/dz & \alpha(z) \\ \beta(z) & d/dz \end{pmatrix}\begin{pmatrix} \hat{P} \\ \hat{Q} \end{pmatrix}=0. \tag{A3, 4}$$

From now on circumflexes will be dropped on $\hat{P}_{\rm T}$ and $\hat{Q}$. We choose rigid boundaries at $z=z_1$, $z=z_2$, that is $Q(z_1)=Q(z_2)=0$. Then eliminating $P$ from (A4) we have, subject to these boundary conditions

$$\frac{d}{dz}\left\{\frac{1}{\beta}\frac{dQ}{dz}\right\}-\alpha Q=0. \tag{18}$$

The crucial quantity in formulating the stability theorem in I was the complex normal mode wave energy flux $-i\bar{P}(k, \omega; z)Q(k, \omega; z)$ (suppressing a time factor). Here we define the flux $\mathfrak{F}(k, \omega; z)$ by $\mathfrak{F}(k, \omega; z)=-i\omega P\bar{Q}$. This is done for analytic convenience; there exists an obvious symmetry in these two expressions which without change in the results entitles this to be done (see I for further details). From equation (A4) we find that

$$\mathfrak{F}(k, \omega; z)=i\omega\bar{Q}Q'/\beta$$

and (19)

$$\frac{d\mathfrak{F}}{dz}(k, \omega; z)=i\omega\left\{\alpha|Q|^2+\frac{|Q'|^2}{\beta}\right\}$$

where equation (18) has been used, and a factor $\exp(-2\int\alpha_1\, dx)$ has been suppressed for convenience on the left-hand side; because of the nature of the boundary conditions the right-hand side is unaffected.

We can obtain the result below by making rather long algebraic manipulations as in I, but we note from equations (A3 and 4) that

$$|P|^2 = |Q'|^2/|\beta|^2 \qquad \text{and } |P'|^2/|\alpha|^2 = |Q|^2.$$

So we may proceed directly to the form (6.25) of paper I to obtain the result

$$[\{c_r - \tfrac{1}{2}(a+b)\}^2 + c_i^2 - \tfrac{1}{4}(a-b)^2]\langle\Phi\rangle + A \leqslant 0 \tag{20}$$

where now

$$\Phi = \left(1 + \frac{\Omega_k^2\Omega_g^2}{W}\right)|Q|^2 + \frac{k^2c_0^4}{W|\beta|^2\tilde{a}_0^4}|Q'|^2 \geqslant 0$$

and

$$A = \left\langle k^{-2}n_k^2|Q|^2 + \left\{\tilde{a}_0^2 + \frac{\Omega_k^2k^2c_0^4}{W}\right\}\frac{|Q'|^2}{\tilde{a}_0^4|\beta|^2}\right\rangle \geqslant 0$$

where $W = |\Omega^2 - \Omega_k^2|^2$, $\max_z\{U_0\} = b$, $\min_z\{U_0\} = a$, $c_0(z)$ is the sound speed and $c = \omega/k = c_r + ic_i$ is the complex phase velocity. Result (20) is a semicircle theorem of the type we have already discussed, and clearly can be improved on by consideration, in particular, of lower bounds on $A$ in terms of $\Phi$. We can define an effective radius $R$ where

$$R^2 = \tfrac{1}{4}(a-b)^2 - A/\langle\Phi\rangle \tag{21}$$

and this elucidates the stabilising effects of various parameters of interest (see I).

For the case of incompressible non-magnetic shear flow, Kochar and Jain (1979) establish a lower bound on the term corresponding to $A$ above which enables them to establish a *semi-ellipse* theorem, incorporating the effect of stratification in a more natural fashion. Unfortunately, for the compressible magnetogravity problem here considered, it appears much more difficult to accomplish this owing to the complexity of $A$ and its dependence on $\omega$ and $U_0$. Nevertheless we achieve some measure of success in this direction by obtaining a 'semi-dumbell' theorem which, at worst (given a certain constraint) is equivalent to the semicircle theorem given above, and is generally stronger.

Firstly however, to compare the effective radius derived from the stably stratified magnetogravity problem in the previous section with that implicit in the result (21) we take the limit $\gamma(=c_0^2\rho_0/p_0) \to \infty$ to yield the results for incompressible flow, namely

$$A = \langle k^{-2}n_k^2|Q|^2 + a_0^2|Q'|^2\rangle \qquad \Phi = |Q|^2 + k^{-2}|Q'|^2;$$

now $Q = \rho_0^{1/2}\xi$ for a Boussinesq fluid (see Appendix) so

$$R^2 = \tfrac{1}{4}(a-b)^2 - \langle\rho_0\{a_0^2\tilde{Q}(\xi) + n_0^2|Q|^2\}\rangle/\langle\tilde{Q}(\xi)\rangle$$

which is easily seen to be equivalent to the result already obtained. Thus it appears that it is primarily the effect of compressibility which induces $\omega$-dependence on $R$. In the next section we make the compressible result rather stronger for a certain class of magnetoatmospheric disturbances.

## 5. A semi-dumbell theorem

Let us decompose the integral $A$ into the following parts:

$$A = \langle k^{-2}(\chi_1\phi + \chi_2\phi_2)\rangle$$

where

$$\phi_1 = \left(1 + \frac{\Omega_k^2\Omega_g^2}{W}\right)|Q|^2, \qquad \Phi_2 = \frac{k^2 c_0^4 |Q'|^2}{W\tilde{a}_0^4|\beta|^2}$$

and

$$\chi_1 = \frac{n_k^2}{1 + W^{-1}\Omega_k^2\Omega_g^2}, \qquad \chi_2 = \frac{a_0^2 W}{k^2 c_0^4} + \Omega_k^2.$$

Using the inequality $W \geqslant 4\omega_i^2(\omega_r - kU_0)^2$ we have

$$\frac{n_k^2 W}{W + \Omega_k^2\Omega_g^2} \geqslant \frac{4\omega_i^2(\omega_r - kU_0)^2 n_k^2}{\Lambda + \Omega_k^2\Omega_g^2}$$

where $\Lambda = \max_{\omega,k,z} W$ and

$$\frac{\tilde{a}_0^2 W}{k^2 c_0^4} + \Omega_k^2 \geqslant \frac{4\omega_i^2(\omega_r - kU_0)^2\tilde{a}_0^2}{k^2 c_0^4}.$$

Therefore $A \geqslant 4c_i^2(c_r - U_0)^2 a_0 c_0^{-4}(\Phi\sqrt{\Phi_1})$ where $c_i = \omega_i/k$, $c_r = \omega_r/k$ and

$$K(\omega, k; z) = \frac{n_k^2 k^2 c_0^4}{(\Lambda + \Omega_k^2\Omega_g^2)\tilde{a}_0^2} - 1.$$

For future reference we introduce the parameter

$$\eta(w, k; z) = \left[n_k^2 - \frac{g^2}{\tilde{a}_0^2}\left(\frac{a_0}{c_0}\right)^4\right]\frac{c_0^4}{k^2\tilde{a}_0^2}$$

so that

$$K = \frac{\eta k^4 - \Lambda}{\Lambda + \Omega_g^2\Omega_k^2}.$$

Defining $\tilde{c}_r = c_r - \frac{1}{2}(a + b)$ we have

$$\langle (c_r - U_0)^2\tilde{a}_0^2 c_0^{-4}\Phi\rangle \geqslant \tfrac{1}{4}(|c_r| + \tfrac{1}{2}(a - b))^2\sigma\langle\Phi\rangle$$

where $\sigma = 4\min(\tilde{a}_0^2/c_0^4) > 0$. Hence $A \geqslant c_i^2\sigma(|c_r| + \frac{1}{2}(a - b))^2(\langle\Phi\rangle + \langle K\Phi_1\rangle)$ and inequality (20) takes the form

$$[|\tilde{c}_r|^2 + c_i^2 + \sigma c_i^2[|\tilde{c}_r| + \tfrac{1}{2}(a - b)]^2 - \tfrac{1}{4}(a - b)^2]\langle\Phi\rangle + \sigma c_i^2[|\tilde{c}_r| + \tfrac{1}{2}(a - b)]^2\langle K\Phi_1\rangle \leqslant 0. \qquad (22)$$

This result is conditional on the sign of $K$: if $K \geqslant 0$ we can state at once that the complex eigenvalues of the problem are contained within a curve defined by (at most)

$$|\tilde{c}_r|^2 + c_i^2 + \sigma c_i^2(|\tilde{c}_r| + \tfrac{1}{2}(a - b)^2 - \tfrac{1}{4}(a - b)^2 \leqslant 0 \qquad (23)$$

(semi-dumbell theorem—see figure 2.) The symmetry in $\tilde{c}_r$ is apparent since the inequality involves $|\tilde{c}_r|$ only.

As will be seen below, this represents an improvement on the standard semicircle result, even compared with equation (21) (which has a reduced semicircle) because we

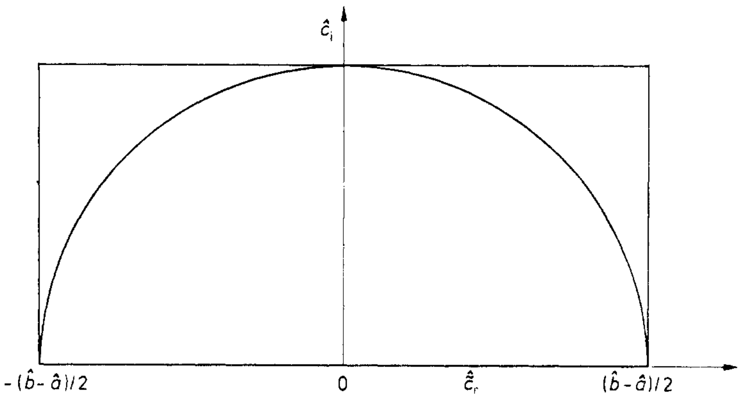


**Figure 1.** Schematic illustration of the rectangle and semicircle theorems for magnetoatmospheric flow in the limit $\gamma \to \infty$ (magnetogravity flow), $\hat{a} \leq \hat{U}_0(z) \leq \hat{b}$, in the complex $\hat{c}$ plane.

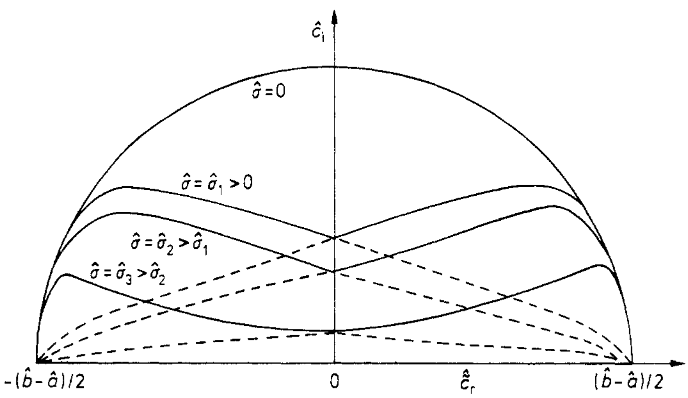


**Figure 2.** Schematic illustration of the 'semi-dumbell' theorem for magnetoatmospheric flow ($\gamma < \infty$), $\hat{a} \leq \hat{U}_0(z) \leq \hat{b}$, in the complex $\hat{c}$ plane for various values of $\hat{\sigma}$. The limiting case $\hat{\sigma} \to 0$ recovers the semicircle theorem.

know the shape of the maximal boundary. However, we are clearly neglecting some information on $c$ contained in the second term and so the result for $K \geq 0$, while a good improvement is clearly not optimal. The case $K<0$ is less helpful, as might be expected, and implies that the eigenvalues do not necessarily lie within the stated curve.

The question arises as to under what conditions $K \geq 0$. Clearly, for $\eta > 0$ a necessary and sufficient condition for $K \geq 0$ is that $\eta k^4 \geq \Lambda$. A crude upper bound on $\Lambda$ may be obtained from the inequalities $c_i^2+(c_r-U_0)^2 \leq (b-a)^2$, $c_i^2 \leq \frac{1}{4}(b-a)^2$, $\max[a_0^2c_0^2/\tilde{a}_0^2] \leq \max(a_0^2, c_0^2) = V_0^2$ say, and $\Omega_k^2 \leq V_0^2k^2$. The first inequality comes from the fact that since both $c$ and $U_0$ are inside the circle or on its perimeter, they cannot be further apart than a diameter, that is $|c-U_0| \leq b-a$. These inequalities enable us to write

$$\Lambda \leq k^4\{[V_0^2+\tfrac{1}{4}(b-a)^2]^2+\tfrac{15}{16}(b-a)^4\} \equiv \eta_0 k^4 \tag{24}$$

which is an upper bound on $\Lambda$ indpendent of $\omega$. Then a sufficient condition for $K \geq 0$ is that $\eta > \eta_0$. Note that $\eta < 0$ corresponds to $K < 0$, so before investigating circumstances under which the sufficient condition $\eta > \eta_0$ is satisfied it is necessary to determine the sign of $\eta$. From its definition we see that

$$\eta > 0 \qquad \text{provided } n_0^2 + a_0^2 k^2 > \frac{g^2}{a_0^2 + c_0^2}\left(\frac{a_0}{c_0}\right)^4. \tag{25}$$

Sufficient conditions for $\eta$ to be positive (irrespective of the value of $a_0/c_0$) are both (i) $\eta_0^2 > 0 (H_s > H)$ and (ii) $H_s k > 1$. Hence $\eta$ will certainly be positive in those stably-stratified regions of the magnetoatmosphere for sufficiently short wavelengths. An alternative sufficient condition, independent of the value of $k$, is that

$$\frac{g}{H} - \frac{g}{H_s} > \frac{g^2}{\tilde{a}_0^2}\left(\frac{a_0}{c_0}\right)^4. \tag{iii}$$

For the case of an isothermal atmosphere and uniform horizontal magnetic field $\eta_0^2 = (\gamma - 1) g^2 / c_0^2$ and the inequality becomes $c_0^4 + a_0^2 c_0^2 - (\gamma - 1)^{-1} a_0^4 > 0$, which is satisfied provided

$$\frac{a_0^2}{c_0^2} < 2\left[\left(\frac{\gamma + 3}{\gamma - 1}\right)^{1/2} - 1\right]^{-1}. \tag{iii}$$

For $\gamma$ in the range $[1, \frac{5}{3}]$ the corresponding upper bounds on $a_0^2/c_0^2$ from this expression lie in the range $[0, 2/(\sqrt{7} - 1)]$.

It is important to note in this example that since $a_0^2$ increases exponentially with altitude (since the magnetic field strength is constant) the maximum value of $a_0^2/c_0^2$ in the region of interest must satisfy (iii) for the sufficiency condition to be valid. This condition, when satisfied, places the range of $a_0/c_0$ in the photospheric and low chromospheric regions of the solar amospheric, that is in regions for which $a_0^2 \leqslant c_0^2$. The flows observed in these regions (Müller 1973) are subsonic, whereas corresponding flows in the coronal regions may reach supersonic velocities.

The sufficient conditions (i) and (ii) for $\eta > 0$, taken together, imply physically that for small enough wavelength and stable enough density stratification, less energy is available for destabilisation of the system. Condition (iii) deals with density profiles that may be even more stably stratified, with $\eta > 0$ for all wavelengths when this is satisfied.

Under what circumstances is $\eta > \eta_0$, that is

$$V_0^4 + \tfrac{1}{2}(b - a)^2 V_0^2 + (b - a)^4 < \eta? \tag{26}$$

Note that when $\eta > 0$, $\eta < n_k^2 c_0^4 / k^2 \tilde{a}_0^2$. Hence a sufficient condition for $K > 0$ is, in terms of an upper bound on $k^2$:

$$k^2 < c_0^4 n_0^2 \{(a_0^2 + c_0^2)[V_0^4 + \tfrac{1}{2}(b - a)^2 V_0^2 + (b - a)^4] - a_0^2 c_0^4\}^{-1} = k_{\max}^2(z). \tag{27}$$

For typical photospheric and low chromospheric values of these parameters, characteristic of penumbral flows (Müller 1973, $(a_0^2 \sim c_0^2 \sim (40\ \mathrm{km\ s^{-1}})^2$, $(b - a) \sim 5\ \mathrm{km\ s^{-1}}$, $n_0^2 \sim 3 \times 10^{-5}\ \mathrm{s^{-2}}$) the inequality (27) is in general incompatible with sufficient condition (ii), namely $k^2 > H_s^{-2} = k_{\min}^2$. Thus for our purposes the sufficient condition (iii) is more useful in conjunction with (27). Thus provided at a point $a_0^2/c_0^2$ is sufficiently small, and the wavelength is sufficiently large, $K$ will be positive at that point. Over a range of altitude we must take account of the fact that both $a_0^2$ and $k_{\max}^2$ are altitude dependent.

This will mean that over a range of altitude $D$ say, sufficient conditions for $K>0$ and hence the semi-dumbell theorem to apply are

$$\max_D\left(\frac{a_0^2}{c_0^2}\right)<2\left[\left(\frac{\gamma+3}{\gamma-1}\right)^{1/2}-1\right]^{-1} \qquad \text{and} \quad k^2<\min_D(k_{\max}^2) \qquad (28), (29)$$

assuming $\gamma$ is constant in $D$. (In practice it is somewhat variable in the solar atmosphere but (28) remains valid in this case if the right-hand side is taken to be $\min_D 2[(\gamma+3/\gamma-1)^{1/2}-1]^{-1}$).

It is not the purpose of this paper to investigate the detailed application of the conditions (28) and (29) in the context of the solar problems, but it is appropriate to consider here the above typical values of parameters and the corresponding value of $k_{\max}^2$. Thus for $\gamma=\frac{5}{3}$ condition (iii) is satisfied (if $\gamma<1\cdot5$ then $a_0^2/c_0^2$ must be $<1$) for the above-mentioned atmospheric parameters and we find $k_{\max}^2\approx 0\cdot3\times10^{-6}\ \text{km}^{-2}$. Hence a typical length scale of horizontal variation $L=k^{-1}$ must be such that $L>k_{\max}^{-1}\approx$ 2000 km, which is of the order of one–two granule sizes. Thus for these parameters, provided the typical length scale of horizontal variation associated with the mode exceeds about 2 000 km, then $K>0$ and the semi-dumbell theorem holds. This means (see figure 2) that while the mode may be unstable, the value of $c_i$ corresponding to it is reduced compared with the semicircle case ($K<0$) for the same value of $c_r$. Since solutions of the form $\xi(x, z, t)\sim\varphi(k, \omega; z)\exp(\mathrm{i}k(x-ct)]$, are being sought, the growth of an initial disturbance in time is given by $\exp(kc_it)\exp(-\mathrm{i}kc_rt)$. Thus if the semi-dumbell theorem is valid we may conclude that the maximum possible growth rate of an unstable mode with given $c_r$ is less (possibly much less) than the maximum possible growth rate when $K<0$. This is consistent with the known result that while disturbances with the largest wavelengths are most likely to be unstable, they possess in general the smallest growth rates (see references in Adam (1978a). From (20 and 30) we see that these maximum growth rates are respectively, for $\tilde{c}_r=0$ (this corresponds to the maximum difference between the two results)

$$(c_i)_{\max}^{K<0}=\tfrac{1}{2}(b-a), \qquad \text{and} \quad (c_i)_{\max}^{K>0}=\tfrac{1}{2}(b-a)\{1+\sigma[(b-a)/2]^2\}^{-1/2}$$

(where for $K<0$ we use the semicircle theorem (20)). Once again using the above parameters we find that typically $\sigma_{\max}=\frac{1}{5}$ and so for $\tilde{c}_r=0$, $(c_i)_{\max}^{K<0}=\frac{5}{2}\ \text{km s}^{-1}$, $(c_i)_{\max}^{K>0}=\frac{5}{3}\ \text{km s}^{-1}$, a factor $\frac{2}{3}$ down. If $b-a=10\ \text{km s}^{-1}$, $(c_i)_{\max}^{K<0}=5\ \text{km s}^{-1}$, $(c_i)_{\max}^{K>0}=5/\sqrt{6}\ \text{km s}^{-1}$ a factor of more than two smaller. From the shape of the bounding semi-dumbell curve in figure 2, it is apparent that the discrepancy between the two cases $K<0$ and $K\geqslant0$ decreases as $|\tilde{c}_r|$ increases from zero, so the differences are seen to be largest when $\tilde{c}_r=0$ corresponding to modes with (real) horizontal phase velocity of $\frac{1}{2}(a+b)$, the mean flow velocity. All these comments are valid, of course, provided $k<k_{\max}$ for $K$ to be positive semidefinite.

## 6. Conclusion

Clearly, if $\sigma$ in expression (23) is sufficiently large the shape of the bounding curve for unstable modes will differ significantly from that of the semicircle ($\sigma=0$). Before defining a non-dimensional form of $\sigma$ we write (23) in the form

$$c_i^2\leqslant\frac{\frac{1}{4}(a-b)^2-|\tilde{c}_r|^2}{(1+\sigma[|\tilde{c}_r|+\frac{1}{2}(a-b)]^2)} \qquad (30)$$

so for large $\sigma$ we note again that the value of $c_i$ is considerably smaller than the corresponding value for $\sigma = 0$. This fact, together with the symmetry about the line $c_r = 0$, leads to the characteristic dumbell shape illustrated in figure 2. This schematic illustration is based on equation (30), suitably non-dimensionalised with respect to $c_0$. Thus if $\hat{m} = m/c_0$ etc. then the semi-dumbell theorem becomes ($K \geqslant 0$)

$$|\hat{c}_r|^2 + \hat{c}_i^2 + \hat{\sigma}\hat{c}_i^2(|\hat{c}_r| - \tfrac{1}{2}(\hat{a} - \hat{b}))^2 - \tfrac{1}{4}(\hat{a} - \hat{b})^2 \leqslant 0 \tag{31}$$

where $\hat{\sigma} = 4 \min_z(a_0^2 + c_0^2)/c_0^2 \geqslant 4$.

Thus $\hat{\sigma}$ is a measure of the total effective pressure (magnetic plus kinetic) compared with the kinetic pressure, since

$$\hat{\sigma} = 1 + \frac{a_0^2}{c_0^2} = 1 + \frac{2}{\gamma}\frac{P_{\text{mag}}}{P_{\text{gas}}}$$

Thus the semi-dumbell theorem would appear from this to be most powerful when $\hat{\sigma} \gg 1$, corresponding to solar coronal conditions. However, as we have already noted, under these circumstances the sufficient conditions for the applicability of the theorem are unlikely to hold, and this particular class of modes will only satisfy in general the standard semicircle theorem. However, in photospheric and chromospheric magnetic regions where $4 \leqslant \hat{\sigma} < 10$, say, the theorem, when applicable, is still a very strong one. Preliminary calculations indicate that the theorem is applicable, at least in part, to the problem of Evershed flow in sunspot penumbrae. A fuller discussion of this application is inappropriate here, but a further theoretical problem of interest is to establish a lower upper bound on $\Lambda$ to improve the eigenvalue bounds presented here.

## Appendix

To make this paper as self contained as possible the basic equations and results of paper I (Adam 1978) are briefly stated.

The linearised equations of continuity, motion, induction and adiabatic compressibility reduce to the equations below for the case of a flow $(U_0(z), 0, 0)$ along a magnetic field $(B_0(z), 0, 0)$ in a compressible stratified medium (with $\partial/\partial y \equiv 0$).

$$\left(\frac{\partial}{\partial z} + \frac{g}{c_0^2}\right) p + \rho_0\left(n_0^2 + \frac{\mathrm{d}^2}{\mathrm{d}t^2} + \frac{g a_0^2}{c_0^2}\frac{\partial}{\partial z} - a_0^2 \frac{\partial^2}{\partial x^2}\right)\xi = 0 \tag{A1}$$

$$\left[\frac{1}{c_0^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \frac{\partial^2}{\partial x^2}\right] p + \rho_0 \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left\{\left[1 + \frac{a_0^2}{c_0^2} - a_0^2\frac{\partial^2}{\partial x^2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{-2}\right]\frac{\partial}{\partial z} - \frac{g}{c_0^2}\right\}\xi = 0 \tag{A2}$$

where $(\mathrm{d}/\mathrm{d}t) = (\partial/\partial t) + U_0(\partial/\partial x)$. The Fourier–Laplace transform

$$\begin{pmatrix} \tilde{p} & (k, \omega; z) \\ \tilde{\xi} & (k, \omega; z) \end{pmatrix} = \int_{-\infty}^{\infty} \mathrm{d}x \int_0^{\infty} \mathrm{d}t \begin{pmatrix} p & (x, z, t) \\ \xi & (x, z, t) \end{pmatrix} \exp[-\mathrm{i}(kx - \omega t)]$$

enables us to obtain a coupled set of ordinary differential equations

$$\left(\frac{\mathrm{d}}{\mathrm{d}z} + a_1\right)\rho_0\tilde{p} + \left(a_2 + a_3\frac{\mathrm{d}}{\mathrm{d}z}\right)\rho_0^{1/2}\tilde{\xi} = 0$$

$$a_4\rho_0^{-1/2}\tilde{p} + \left(a_5 + a_6\frac{\mathrm{d}}{\mathrm{d}z}\right)\rho_0^{1/2}\tilde{\xi} = 0$$

where

$$a_1=\frac{g}{c_0^2}-\frac{1}{2H}, \qquad H=\left\{-\frac{1}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}z}\right\}^{-1}, \qquad n_0^2=\frac{-g}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}z}-\frac{-g^2}{c_0^2}$$

$$a_2=n_0^2-\Omega^2+a_0^2k^2+\frac{ga_0^2}{2Hc_0^2}, \qquad \Omega=\omega-kU_0$$

$$a_3=ga_0^2/c_0^2, \qquad a_4=k^2-\Omega^2/c_0^2$$

$$a_5=\Omega^2\left[\frac{g}{c_0^2}-\frac{1}{2H}\left(\frac{a_0^2}{c_0^2}+1\frac{-a_0^2k^2}{\Omega^2}\right)\right] \qquad a_6=-\Omega^2\left(\frac{a_0^2}{c_0^2}+1\frac{-a_0^2k^2}{\Omega^2}\right).$$

Finally, the change of dependent variables

$$P=\rho_0^{-1/2}(z)\exp\left(\int^z \alpha_1(z)\,\mathrm{d}z\right)\tilde{p}$$

$$Q=\rho_0^{1/2}(z)\exp\left(-\int^z \alpha_1(z)\,dz\right)\tilde{\xi}$$

where $\alpha_1=-a_5/a_6$, leads to the operator equation

$$\begin{pmatrix}\mathrm{d}/\mathrm{d}z & \alpha(z)\\ \beta(z) & \mathrm{d}/\mathrm{d}z\end{pmatrix}\begin{pmatrix}P\\ Q\end{pmatrix}=0 \qquad \text{(A3, 4)}$$

where

$$\alpha(z)=\exp\left(2\int^z \alpha_1\,\mathrm{d}z\right)\left(a_2-\frac{a_3a_5}{a_6}\right), \qquad \text{(A5)}$$

$$\beta(z)=\exp\left(-2\int^z \alpha_1\,\mathrm{d}z\right)\frac{a_4}{a_6}. \qquad \text{(A6)}$$

Manipulation of the various quantities for the free boundary problem ($P(z_1)=P(z_2)=0$) leads to the semicircle result

$$[[c_r-\tfrac{1}{2}(a+b)]^2+c_i^2-\tfrac{1}{4}(a-b)^2]\langle\Phi\rangle+A\leqslant 0 \qquad \text{(A7)}$$

if $c_i>0$, where

$$\Phi=\left(1+\frac{\Omega_k^2\Omega_g^2}{W}\right)\frac{|P'|^2}{|\alpha|^2}+\frac{k^2c_0^4}{W\tilde{a}_0^4}|P|^2\geqslant 0$$

$$A=\left\langle k^{-2}\frac{n_k^2}{|\alpha|^2}|P'|^2+\left\{\tilde{a}_0^2+\frac{\Omega_k^2k^2c_0^4}{W}\right\}\frac{|P|^2}{\tilde{a}_0^4}\right\rangle\geqslant 0$$

where

$$\Omega_k^2=\frac{a_0^2c_0^2k^2}{\tilde{a}_0^2}=\frac{c_0^2k^2}{\nu^2} \qquad \text{and} \qquad \Omega_g^2=\frac{g^2a_0^2}{c_0^2\tilde{a}_0^2}=\frac{1}{\nu^2}\frac{g}{H_s}.$$

We have introduced the following length scales, velocities and frequencies

$H_s=c_0^2/g$ (a gravito-acoustic length)

$\tilde{a}_0^2=a_0^2+c_0^2$ (magnetoacoustic hybrid velocity $\tilde{a}_0$)

$\nu=\tilde{a}_0/a_0, \qquad n_k^2=n_0^2+a_0^2k^2 \qquad$ and $\Omega_m^2=\Omega^2-\Omega_k^2$

then $\alpha(z)=n_k^2-\Omega^2(1-\Omega_g^2/\Omega_m^2)$ and $\beta(z)=a_0^2[1+(1-\nu^2)\Omega_k^2/\Omega_m^2]$ apart from the exponential factors indicated in (A5) and (A6). The suppression of these is perfectly valid since the analysis makes use of the vanishing of $P(Q)$ in the free (rigid) boundary value problem, so that the flux vanishes at both $z=z_1$ and $z=z_2$.

Reference is made in the main body of the paper to the limiting forms of the $a_i (i=1, 2, \ldots, 6)$ above for a Boussinesq fluid. This limit is obtained as $\gamma \to \infty$, ignoring terms in $[(1/\rho_0)(\mathrm{d}\rho_0/\mathrm{d}z)]$ unless they are multiplied by the gravitational acceleration $g$. Thus in this approximation, the influence of compressibility is ignored, except insofar as the buoyancy force on an element of fluid moving to a new level is concerned.

Thus $\alpha_1=-a_5/a_6=0$ and $Q=\rho_0^{1/2}\xi$ in particular.

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